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A BANACH SPACE CONTAINING NON-TRIVIAL LIMITED SETS BUT NO NON-TRIVIAL BOUNDING SETS

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ABSTRACT

An example of a Banach space E is given with the following properties: Every bounding set $A \subset E$ (i.e. $f(A)$ is bounded for each holomorphic function $f: E \to \mathbf{C}$) is relatively compact but there are relatively non-compact limited sets A (i.e. $T(A)$ is relatively compact for each bounded linear map $T: E \to c_0$).

Introduction

A subset A of a Banach space E is called limited if $\lim_j \varphi_j(x) = 0$ uniformly for $x \in A$ whenever (φ_j) is a weak* null sequence in E^* and A is called bounding if every entire function in E is bounded on A. Since $\sum_1^\infty \varphi_j^j$ is an entire function if (and only if) (φ_j) is a weak* null sequence, it is an immediate consequence that a bounding set also is limited. It has been an open problem, however the opposite inclusion holds (see [D1]). In this paper an example is given which answers this question in the negative.

In [D2] and [J2] examples of non-trivial (i.e. not relatively compact) bounding sets are given. It is shown in [J2] that the unit ball of a Banach space E, viewed as a subspace of $l^\infty(\Gamma)$ where Γ is a sufficiently big index set, is bounding and limited precisely when E does not contain any isomorphic copy of l_1. On the contrary, [J1] gives that the unit ball of an infinite dimensional Banach space is never limited in the space itself. In this paper a Banach space E containing an isomorphic copy of c_0 is constructed, such that the unit vectors of c_0 is limited (which is shown by arguments close to that in [D2]) but E contains no bounding sets besides the trivial ones. In fact, the class of entire functions generated by continuous lin-

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ear functionals is rich enough to conclude that the bounding sets are relatively compact.

I am very grateful to the referee for allowing me to publish this example. First of all, it does not depend on the Continuum Hypothesis, as my original one did, and, secondly, though the basic ideas are the same, it makes much more clear and effective use of these ideas.

Some notations

For a set Γ:

$\mathcal{P}_\infty(\Gamma)$:= infinite subsets of Γ,
$c_{00}(\Gamma) := \{(a(\gamma):\gamma\in\Gamma)\subset \mathbf{C} \mid \{\gamma : a(\gamma)\neq 0\} \text{ is finite}\}$,
$(e_\gamma:\gamma\in\Gamma)$ usual Hammel-basis of $c_{00}(\Gamma)$
(i.e. $e_\gamma(\gamma)=1$ and $e_\gamma(\gamma')=0$ if $\gamma'\neq\gamma$).

For the Banach space l_∞, $\|\cdot\|_{l_\infty}$ denotes its norm and if $M\subset\mathbf{N}$, $\chi_M\in l_\infty$ is defined by $\chi_M(n)=1$, if $n\in M$, and $\chi_M(n)=0$, if $n\in\mathbf{N}\backslash M$. For $j\in\mathbf{N}$, $\mathrm{proj}_j : l_\infty\to\mathbf{C}$ is defined by $\mathrm{proj}_j(x_i)=x_j$ if $(x_i)\in l_\infty$.

Construction of E

Let $\Gamma := \mathcal{P}_\infty(\mathbf{N})\times\mathbf{N}\times\omega_1$, where ω_1 denotes the first uncountable ordinal, and, if $M\in\mathcal{P}_\infty(\mathbf{N})$ and $k\in\mathbf{N}$, we set $\Gamma_{(M,k)} := \{(M,k,\alpha):\alpha\in\omega_1\}$. For each $M\in\mathcal{P}_\infty(\mathbf{N})$ and $k\in\mathbf{N}$ we choose a family $(V_{(M,k,\alpha)}:\alpha<\omega_1)\subset\mathcal{P}_\infty(M)$ satisfying

(1) $$V_{(M,k,\alpha)}\cap V_{(M,k,\beta)} \text{ is finite if } \alpha\neq\beta$$

and we put for $\gamma=(M,k,\alpha)\in\Gamma$

(2) $$f_\gamma := 2^{-k}\chi_{V_{(M,k,\alpha)}},$$

considering f_γ as an element of $l_\infty(\mathbf{N})$. For each $z=\sum_{n\in\mathbf{N}} z_n\cdot e_n+\sum_{\gamma\in\Gamma} z_\gamma e_\gamma\in c_{00}(\mathbf{N}\cup\Gamma)$ we define

(3) $$\|z\|_0 := \sum_{(M,k)\in\mathcal{P}_\infty(\mathbf{N})\times\mathbf{N}} \max\left\{\sup_{\gamma\in\Gamma_{(M,k)}} |z_\gamma|, \frac{1}{k\cdot 2^k}\sum_{\gamma\in\Gamma_{(M,k)}} |z_\gamma|\right\}$$

and

(4) $$\|z\|_\infty := \left\|\sum_{k\in\mathbf{N}} z_k\chi_{\{k\}}+\sum_{\gamma\in\Gamma} z_\gamma f_\gamma\right\|_{l_\infty}.$$

Since $\|z\|_0 \neq 0$ iff $\sum z_\gamma e_\gamma \neq 0$ and $\|z\|_\infty \neq 0$ if $\sum z_\gamma e_\gamma = 0$ and $\sum x_k \chi_{\{k\}} \neq 0$, $\|z\| := \max\{\|z\|_\infty, \|z\|_0\}$, for $z \in c_{00}(\mathbf{N} \cup \Gamma)$, defines a norm. We let E be the completion of $c_{00}(\mathbf{N} \cup \Gamma)$ under $\|\cdot\|$. Furthermore, we define E_∞ to be the closed subspace of l_∞ generated by $\{\chi_{\{k\}}, f_\gamma : k \in \mathbf{N}, \gamma \in \Gamma\}$ and E_0 to be the completion of $c_{00}(\Gamma)$ under $\|\cdot\|_0$. We note that E_0 is the l_1-direct-sum of spaces $E_{(M,k)}$ where, for $(M,k) \in \mathcal{P}_\infty(\mathbf{N}) \times \mathbf{N}$, $E_{(M,k)}$ is the closed subspace of E_0 generated by $\{e_\gamma : \gamma \in \Gamma_{(M,k)}\}$. Also, each $E_{(M,k)}$ is isomorphic to $l_1(\Gamma_{(M,k)})$. This implies, in particular, that E_0 has the Schur property.

The mapping $c_0 \ni (x_n) \mapsto \sum x_n e_n \in E$ defines an isometric embedding of c_0 into E. We shall always mean the range of this isometry if we speak of c_0 as a subspace of E.

For $z = \sum_{j \in \mathbf{N}} z_j e_j + \sum_{\gamma \in \Gamma} z_\gamma e_\gamma \in c_{00}(\mathbf{N} \cup \Gamma)$ and $n \in \mathbf{N}$ we set

$$ {}^n z := \sum_{j>n} z_j e_j + \sum_{M \in \mathcal{P}_\infty(\mathbf{N}), k \in \mathbf{N}} \sum_{\gamma \in \Gamma_{(M,k)}} z_\gamma \left(e_\gamma - \sum_{j \le n, j \in V_\gamma} 2^{-k} e_j \right). \tag{5} $$

If we let $T : E \to E_\infty$ be the operator defined by $T(e_k) = \chi_{\{k\}}$ and $T(e_\gamma) = f_\gamma$, it follows from the definition (2) of f_γ and from (5) that

$$ T({}^n z) = T(z) \cdot \chi_{\{n+1, n+2, \dots\}}, \qquad \text{if } z \in c_{00}(\mathbf{N} \cup \Gamma). \tag{6} $$

By condition (1), it is possible to find for $z \in c_{00}(\mathbf{N} \cup \Gamma)$ an $n_0 \in \mathbf{N}$ so that $V_\gamma \cap V_{\gamma'} \subset \{1, \dots, n_0\}$ whenever there is a $k \in \mathbf{N}$ and an $M \in \mathcal{P}_\infty(\mathbf{N})$ with $\gamma, \gamma' \in \Gamma_{(M,k)}$, $\gamma \neq \gamma'$, and $z_\gamma, z_{\gamma'} \neq 0$.

This implies that for all $z \in c_{00}(\mathbf{N} \cup \Gamma)$ there exists an n_0 so that

$$ \|{}^n z\| = \|z\|_0 \qquad \text{if } n \geq n_0, \tag{7} $$

and that n_0 only depends on the set $\{k \in \mathbf{N} : z_k \neq 0\} \cup \{\gamma \in \Gamma : z_\gamma \neq 0\}$. For the quotient space E/c_0 and its norm $\|\ \|_{E/c_0}$ we deduce that

$$ \|z\|_{E/c_0} \leq \liminf_{n \to \infty} \|{}^n z\| = \|z\|_0 \leq \|z\|_{E/c_0} $$

if $z \in c_{00}(\mathbf{N} \cup \Gamma)$ and thus that

$$ \|z\|_{E/c_0} = \|z\|_0 \qquad \text{if } z \in E. \tag{8} $$

Since E_0 has the Schur property, this leads to the following dichonomy of bounded non-relatively compact subsets of E. Let $A \subset E$ be bounded and non-relatively compact, then:

Either its image under the quotient mapping $E \to E/c_0 \cong E_0$ is also not relatively compact. Then its image, and thus A itself, contains a sequence equivalent to the l_1-unit vector basis.

Or this is not the case. Then there exists a compact set $C \subset E$ and a bounded set $B \subset c_0$ so that $A \subset C + B$. Indeed, choose a compact $C \subset E$ with $T(C) \supset T(A)$ and $B := 2(\sup_{x \in A} \|x\|)\mathrm{Ball}(c_0)$ (such a C exists because every compact set of E/c_0 is in the absolute convex hull of a norm-null sequence).

About limited sets in E

LEMMA 1. *The unit vector basis of c_0 (e_k) is limited in E and each limited set in E is contained in some $B + C$ for some bounded set $B \subset c_0$ and some compact set $C \subset E$.*

PROOF. In order to prove the second statement, let $A \subset E$ be limited (thus bounded). Then its image under the quotient mapping $T: E \to E/c_0 \cong E_0$ is also limited. Since sequences which are equivalent to the unit vector basis of l_1 are not limited [BD], $T(A)$ must be relatively compact. Since A is bounded this yields the assertion, by the above-stated dichonomy.

In order to prove the first statement, let $(\varphi_n) \subset E^*$ be such that $C := \sup_{n \in \mathbf{N}} \|\varphi_n\| < \infty$ and $\varphi_n(e_{m_n}) = 1$, $n \in \mathbf{N}$, for a subsequence (m_n) of $\mathbf{N}$. We have to show that (φ_n) is not ω^* convergent to zero. We may assume that (φ_n) is pointwise null on c_0. Since (e_{m_n}) is weakly null, for each $k \in \mathbf{N}$ it follows that $\lim_{n\to\infty} \varphi_n(e_{m_k}) = \lim_{n\to\infty} \varphi_k(e_{m_n}) = 0$. By a standard perturbation and subsequence argument we can assume that

$$\varphi_k(e_{m_n}) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

Let $M := \{m_n : n \in \mathbf{N}\}$ and fix a $k \in \mathbf{N}$ with $k \geq C$. We claim that for each $j \in \mathbf{N}$ the set $F_j := \{\gamma \in \Gamma_{(M,k)} \mid \liminf_{n\to\infty} |\varphi_j({}^n e_\gamma)| > 2^{-k-1}\}$ has less than $k \cdot 2^k$ elements (${}^n z$ for $z \in c_{00}(\mathbf{N} \cup \Gamma)$ and $n \in \mathbf{N}$ was defined in (5) of the previous section). Otherwise there are distinct $\gamma_1, \gamma_2, \ldots, \gamma_{k \cdot 2^k} \in \Gamma_{(M,k)}$ so that

$$\liminf_{n\to\infty} \varphi_j\left(\sum_{i=1}^{k\cdot 2^k} \mathrm{sign}(\varphi_j({}^n e_{\gamma_i})) \cdot {}^n e_{\gamma_i}\right) \geq 2^{-k-1} \cdot k \cdot 2^k > 2C.$$

Since by (7) of the previous section we have $\lim_{n\to\infty} \|\sum_{i=1}^{k\cdot 2^k} x_i {}^n e_{\gamma_i}\| = \|\sum_{i=1}^{k\cdot 2^k} x_i {}^n e_{\gamma_i}\|_0 = 1$ for $x_1, \ldots, x_{k\cdot 2^k} \in \mathbf{C}$ with $|x_i| = 1$ and since $\|\varphi_j\| \leq C$, this leads to a contradiction.

Now choose $\gamma \in \Gamma_{(M,k)} \setminus \bigcup_{j \in \mathbf{N}} F_j$ (which exists since $\Gamma_{(M,k)}$ is uncountable). We thus conclude for $m_j \in V_\gamma$ and all $n \in \mathbf{N}$ with $n \geq m_j$ that

$$\varphi_j(e_\gamma) = \varphi_j\left(\sum_{\substack{m\le n\\ m\in V_\gamma}} 2^{-k}e_m\right) + \varphi_j({}^ne_\gamma)$$
$$= 2^{-k} + \varphi_j({}^ne_\gamma);$$

since $|\liminf_{n\to\infty}\varphi_j({}^ne_\gamma)| < 2^{-k-1}$, $(\varphi_j(e_\gamma))$ cannot converge to zero.

All bounding subsets of E are relatively compact

All bounding sets are limited and, by Lemma 1, all limited subsets of E are contained in some $B + C$, where B is a bounded subset of c_0 and C is relatively compact. Thus, because the sum of two bounding sets is still bounding [D1, p. 176, Cor.4.24], we need only show that a non-compact bounded set $B \subset c_0$ is not bounding in E.

For this we construct a sequence (p_j) of polynomials on E having the following properties:

(a) For $j \in \mathbf{N}$, p_j is a product of $s_j := 2^j - 1$ continuous linear functionals.
(b) For $j \in \mathbf{N}$ and $x = \sum x_n \cdot e_n \in c_0$, it follows that $p_j(x) = p_j(x_j\cdot e_j) = x_j^{s_j}$.
(c) For every $z \in E$ and $\epsilon > 0$, there is a $\delta > 0$ and a $j = j(z,\epsilon) \in \mathbf{N}$ so that, for all $i \ge j$,

$$|p_i(z+h)| < \epsilon^{s_i} \qquad \text{if } h\in E \quad\text{and}\quad \|h\| < \delta.$$

This will be enough to show that a non-compact bounded subset B of c_0 is not bounding. Indeed, for such a set B we find a sequence $(b^j) \subset B$, $b^j = \sum b_i^j e_i$ and a sequence of increasing integers (n_j) so that $\inf_{j\in\mathbf{N}}|b_{n_j}^j| = C > 0$. From (c) we conclude that $f(z) := \sum_{j=1}^\infty p_{n_j}(z)$, $z \in E$, is locally bounded and, thus, holomorphic. On the other hand, (b) implies that $\liminf_{j\to\infty}\sup_{b\in B}|p_{n_j}(b)|^{1/s_{n_j}} \ge C$; by [D1, p. 173, Cor.4.19] this implies that B is not bounding.

Construction of (p_j)

For $j, n \in \mathbf{N}$ with $n \le j$ and $\gamma \in \Gamma$ put

$$\varphi_{j,n}(e_\gamma) := \begin{cases} 2^{-k} & \text{if } j \in V_\gamma \text{ and } \gamma \in \Gamma_{(M,k)} \text{ for some } M \in \mathcal{P}_\infty(\mathbf{N}) \\ & \text{and some } k \le 2^{n^2} \\ 0 & \text{else} \end{cases}$$

and for $i \in \mathbf{N}$

$$\varphi_{j,n}(e_i) = 0.$$

Since for $z = \sum_{i\in\mathbf{N}} z_i e_i + \sum_{\gamma\in\Gamma} z_\gamma e_\gamma \in c_{00}(\mathbf{N}\cup\Gamma)$ we have

$$\begin{aligned}\left|\sum_{i\in\mathbf{N}} z_i\varphi_{j,n}(e_i) + \sum_{\gamma\in\Gamma} z_\gamma\varphi_{j,n}(e_\gamma)\right| &\le \sum_{M\in\mathcal{P}_\infty(\mathbf{N})}\sum_{k=1}^{2^{n^2}} 2^{-k}\sum_{\substack{\gamma\in\Gamma_{(M,k)}\\ j\in V_\gamma}} |z_\gamma| \\ &\le 2^{n^2}\sum_{M\in\mathcal{P}_\infty(\mathbf{N})}\sum_{k=1}^{2^{n^2}}\frac{1}{2^k\cdot k}\sum_{\gamma\in\Gamma_{(M,k)}}|z_\gamma| \\ &\le 2^{n^2}\|z\|_0 \le 2^{n^2}\|z\|,\end{aligned}$$

it follows that $\varphi_{j,n}$ can be extended to a bounded linear functional on E still denoted by $\varphi_{j,n}$. We put $\psi_{j,n} := \mathrm{proj}_j - \varphi_{j,n}$ (where proj_j acts on E_∞).

Note that for each $\gamma\in\Gamma$

$$\psi_{j,n}(e_\gamma) = \begin{cases} 2^{-k}\chi_{V_\gamma}(j) & \text{if } \gamma\in\bigcup_{M\in\mathcal{P}_\infty(\mathbf{N})}\Gamma_{(M,k)} \text{ for some } k > 2^{n^2}, \\ 0 & \text{else.}\end{cases} \tag{9}$$

Finally, we put for $j\in\mathbf{N}$ and $z\in E$

$$p_j(z) := \prod_{n=1}^{j} (\psi_{j,n}(z))^{2^{j-n}}.$$

Since $\sum_{n=1}^{j} 2^{j-n} = 2^j - 1 = s_j$ and $\psi_{j,n}|_{c_0} = \mathrm{proj}_j|_{c_0}$, (a) and (b) are satisfied. In order to show (c) let $z\in E$ and $\epsilon > 0$. Set $K := \sum_{n=1}^{\infty} 2^{-n}n^2$ and $\epsilon' := \epsilon/3\cdot 2^{K+1}(\|z\|+1)$. Then choose $n_0\in\mathbf{N}$ so that $\|z\|^{2^{-n_0}}2^{-2^{n_0^2}\cdot 2^{-n_0}} < \epsilon'$ and $\delta > 0$ so that $3\cdot 2^{n_0^2}\delta < (\epsilon')^{2^{n_0}}$. In order to choose $j_0 = j_0(z,\epsilon)$ as desired in (c) we first choose $\tilde z = \sum z_n e_n + \sum z_\gamma e_\gamma \in c_{00}(\mathbf{N}\cup\Gamma)$ with $\|\tilde z - z\| < \delta$ and $\|\tilde z\| = \|z\|$. By the property (1) of $(V_\gamma : \gamma\in\Gamma)$ we find $j_0 \ge n_0$, so that for each $M\in\mathcal{P}_\infty(\mathbf{N})$ and $k\in\mathbf{N}$ it follows that $V_\gamma\cap V_{\gamma'}\subset\{1,2,\ldots,j_0-1\}$ whenever $\gamma\ne\gamma'\in\Gamma_{(M,k)}$ and $z_\gamma, z_{\gamma'}\ne 0$. From (9) we deduce for $j\ge j_0$ and $n\le j$ that

$$\begin{aligned}|\psi_{j,n}(\tilde z)| &= \left|\sum_{k>2^{n^2}}\sum_{M\in\mathcal{P}_\infty(\mathbf{N})} 2^{-k}\sum_{\gamma\in\Gamma_{(M,k)}} z_\gamma\chi_{V_\gamma}(j)\right| \\ &\le 2^{-2^{n^2}}\sum_{k>2^{n^2}}\sum_{M\in\mathcal{P}_\infty(\mathbf{N})}\sup_{\gamma\in\Gamma_{(M,k)}}|z_\gamma| \\ &\le 2^{-2^{n^2}}\|\tilde z\|_0 \le 2^{-2^{n^2}}\|z\|.\end{aligned} \tag{10}$$

Thus, we conclude for $h \in E$, $\|h\| \leq \delta$ and $j \geq j_0$ that

$$
\begin{aligned}
|p_j(z+h)|^{1/s_j} &\leq \prod_{n=1}^{j} (|\psi_{j,n}(\tilde{z})| + 2\delta\|\psi_{j,n}\|)^{2^{-n}} \\
&\leq \prod_{n=1}^{j} |2^{-2^{n^2}}\|z\| + 2\cdot(2^{n^2}+1)\delta|^{2^{-n}} \\
&\leq (2^{-2^{n_0^2}}\|z\| + 3\cdot 2^{n_0^2}\delta)^{2^{-n_0}} \prod_{\substack{n=1 \\ n\neq n_0}}^{j} ((1+\|z\|)\cdot 2^{n^2}\cdot 3)^{2^{-n}} \\
&\leq (2^{-2^{n_0^2}\cdot 2^{-n_0}}\|z\|^{2^{-n_0}} + (3\cdot 2^{n_0^2}\cdot\delta)^{2^{-n_0}})3\cdot(1+\|z\|)2^K \\
&\leq 2\cdot\epsilon'\cdot 3(1+\|z\|)\cdot 2^K = \epsilon
\end{aligned}
$$

which implies the assertion.

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